

A Note on the Symmetries and Renormalisability of (Quantum) Gravity

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Abstract

We make some remarks on the group of symmetries in gravity; we believe that K-theory and noncommutative geometry inescapably have to play an important role. Furthermore we make some comments and questions on the recent work of Connes and Kreimer on renormalisation, the Riemann-Hilbert correspondence and their relevance to quantum gravity.

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1 Introduction and Motivation

The main reason for the prominent role of Yang-Mills theories in physics is the fact that such *quantum* theories make sense, namely one can extract *finite* answers for physical quantities through a process known as *renormalisation*. The proof of this celebrating fact was given some 30 years ago by G. 't Hooft who has also introduced the method of *dimensional regularisation*

(see for instance [15]). Regularisation is the first step in the 2-step process of renormalisation where one wants to "parametrise the infinities" which appear in quantum field theories; explicitly one wants to calculate divergent integrals of the form

$$\int_0^\infty d^4k F(k)$$

(where k is essentially the momentum). We know that the 3 out of the 4 known interactions in nature are Yang-Mills theories. Thus one can indeed have a meaningful quantum theory for electroweak and strong interactions.

Gravity however, the 4th known interaction in nature, is a different story: although it can be thought of as a gauge theory, it is not of Yang-Mills type since it has a different action and a different gauge group of symmetries. From the early days of the development of quantum field theory (due to Dirac, Schwinger, Dyson, Feynman etc), people knew that gravity suffers from (*incurable perhaps*) divergencies and infinities and all known methods of regularisation which worked in other theories, such as the Pauli-Villars method, the momentum cutoff method or dimensional regularisation, they all brake down in this case. Of course since the days of A. Einstein, a quantum theory of gravity is every self-respectful physicist's dream. It is perhaps surprising the fact that although gravity is the weakest of all interactions and one might expect perturbative methods to work quite well for it, it is the interaction for which all known renormalisation schemes fail. Being optimistic, we shall not call gravity a nonrenormalisable theory, we shall say that it is only "*superficially nonrenormalisable*" in dimension 4 since the upper critical dimension of Newton's constant G is 2 as follows from the relevant Callan-Symanzik equation (see section 4 below). In such cases, which are not at all promising, what can *perhaps* save the day is an elaborate symmetry argument, certain fixed points of the renormalisation group flow or nonperturbative effects. That's the main motivation for this piece of work.

Aside: We take the point of view that quantum gravity-which is currently an elusive theory-*should exist*; the argument in favour of its existence goes as follows (the original argument we think was due to P.A.M. Dirac): let us consider Einstein's classical field equations which describe gravity (we assume no cosmological constant and we use physical units, i.e. we set the speed of light and Planck's constant equal to one):

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

In the above equation, G denotes Newton's constant, $T_{\mu\nu}$ denotes the energy-momentum tensor and $G_{\mu\nu}$ denotes the Einstein tensor which is equal, by definition, to $G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$, where $g_{\mu\nu}$ is the Riemannian metric, $R_{\mu\nu}$ is the Ricci curvature tensor and R is the scalar curvature. One can see clearly that the RHS of the above equation, namely the energy-momentum tensor, contains mass and energy coming from the other two interactions in nature (electroweak and strong); mass for instance, consists primarily of quarks and leptons (these are both fermions); one also has the massive carriers of the electroweak force, the W and Z bosons; they all acquire mass through the Higgs mechanism (the Higgs boson—the carriers of the strong force and electromagnetism, namely the gluons and the photons, are massless). We know that these interactions (strong and electroweak) are quantized and hence the RHS of the equation contains *quantized quantities*. So for *consistency* of the equations, the LHS, which encodes geometry, *should also be quantized*.

[Comment: one may argue that the LHS may remain classical while the RHS may involve the *average value* of an operator; however such a theory will not be essentially different from classical general relativity and probably not qualified to be called quantum gravity, what we have in mind is Ehrenfest Theorem from Quantum Mechanics. We think of the above field equations as describing, in the quantum level, an actual equality between operators].

2 Gravity and Yang-Mills Theories

Let us elaborate more on the two differences between gravity and Yang-Mills theories (like the strong and the electroweak forces) at the *classical level*. The first difference is the action: in Yang-Mills theories we start with a 4-dim (pseudo) Riemannian manifold M representing spacetime along with a structure Lie group G (say G is some $SU(N)$; to be phenomenologically correct, $G = SU(3)$ for the strong force and $G = U(2)$ for the electroweak force) representing internal symmetries; we thus construct a principal G -bundle over M whose total space P gives the internal space of the theory; we pick a connection A on the bundle P (which represents the gauge potential) with curvature (field strength) $F := d_A A := dA - \frac{1}{2}[A, A]$ where d_A denotes the exterior covariant derivative w.r.t. the connection 1-form A . Then the

(pure) Yang-Mills action reads (ignoring constants)

$$I = \int_M F \wedge *F$$

where “ $*$ ” denotes the Hodge dual which is defined using the Riemannian metric. The group of (internal) gauge transformations is the infinite dim Lie group of bundle automorphisms, denoted B , covering the identity map on the base manifold—sometimes these are called *strong* bundle automorphisms, (or equivalently $B = Maps(M \rightarrow G)$) (see [2]). The Euler-Lagrange equations read

$$d_A * F = \text{“source”}$$

The above equations state the deep geometric fact that *“the curvature of the internal space is caused by the existence of the relevant charges”*. Similarly the corresponding monopoles are singular points where the Bianchi identity fails.

Gravity is different: the (Einstein-Hilbert) action reads

$$I = \int_M R$$

where R denotes the scalar curvature of the Levi-Civita connection defined via the metric. The corresponding Euler-Lagrange equations are Einstein’s equations which (ignoring constants) equate the Einstein tensor with the energy-momentum tensor. These equations are different (but in similar spirit) from the Yang-Mills equations: the internal space is spacetime itself (or its tangent bundle to be more precise) and the relevant charge for the gravitational interaction is mass. Einstein’s equations then tell us qualitatively that *“it is not only mass (ie the relevant charges) which curve the internal space but there is additional curvature coming from the intrinsic geometry of the spacetime manifold itself”*. As about the group of (spacetime) gauge transformations, this is the infinite dim Lie group of local diffeomorphisms $Diff(M)$ of M .

Thus the total group of symmetries, denoted T , is the semi-direct product $T = B \times Diff(M)$. The situation is summarised by the following exact sequence of groups:

$$1 \rightarrow B \rightarrow T \rightarrow Diff(M) \rightarrow 1$$

Clearly in order to unify strong and electroweak forces we should take B to be the group of strong bundle maps with structure group $G = U(2) \times SU(3)$.

If one wishes to build a unifying theory of all interactions, there are two obvious ways to proceed: One can either try to see if there is a "space" \tilde{M} such that $T = Diff(\tilde{M})$. This means that at least as far as symmetries are concerned, we would like to make the would-be unified theory actually "look like" a "gravity" theory on a new spacetime manifold \tilde{M} . This approach was adopted by Connes et al (see [3]) and it is useful if one wants to use the quantum theory in order to reveal the deep underlying "quantum" geometry of spacetime; by following this approach one ends up with the Connes-Lott model and its variations (the double-sheeted spacetime), namely the new spacetime \tilde{M} is a *noncommutative* space (ie a space whose algebra of coordinate functions is noncommutative) where the metric is given by the inverse of the Dirac operator (Dirac propagator) or the Schwinger-Dyson propagator used more recently in [7]. In noncommutative geometry one replaces the group $Diff(M)$ by the automorphism group $Aut(A)$ of a noncommutative algebra A . This follows from Gelfand's theorem and from the exact sequence of groups

$$1 \rightarrow Int(A) \rightarrow Aut(A) \rightarrow Out(A)$$

where $Int(A)$ and $Out(A)$ denote the groups of internal and external automorphisms respectively of the algebra A .

The second way is to try to see if there is a suitable "extended" (Lie perhaps) group \tilde{G} which we use in order to construct a principal \tilde{G} -bundle with total space \tilde{P} over ordinary spacetime M such that T equals the Lie group of strong \tilde{G} -bundle automorphisms (i.e. automorphisms of \tilde{P} covering the identity map on the base space of this extended bundle or equivalently $T = Maps(M \rightarrow \tilde{G})$). In other words, in this approach one wants to make the would-be unified theory "look like" a Yang-Mills theory (at least as far as the symmetries are concerned, no mention of the action at this point). *This approach wishes to make use of the crucial advantage of the renormalisability of Yang-Mills theories* and thus one hopes that this would-be unified theory (containing gravity) will eventually be renormalisable.

The first approach works and indeed we have various proposed models and we get information about the underlying spacetime geometry dictated by quantum theory. Yet we get no information about the quantization

of gravity. Concerning the second approach however, it is not even clear whether such a group like \tilde{G} exists at all. We believe that this is one of the motivations behind the development of various supergravity or super Yang-Mills theories. Yet we should be careful here: people in supergravity start by "gauging the Poincare group". For simplification we assume the Riemannian case and hence the Lorentz group becomes $SO(4)$; this is the structure group of the tangent bundle TM of M where by picking a Riemannian metric a reduction of the structure group takes place, i.e. we go from $GL(4; \mathbf{R})$ to $SO(4)$. Yet if one does this, one gets as gauge symmetry group the group of bundle automorphisms of the tangent bundle TM of M which cover the identity map on the base (the strong tangent bundle automorphisms); *clearly, this group is NOT $Diff(M)$, neither does it contain $Diff(M)$.*

So people add *fermionic degrees of freedom (Grassmann variables)* (make use of the Coleman-Mandula theorem) and now the picture starts becoming messy: our understanding is that these grassmann variables are added to the structure group and hence one ends up with a super-Lie group. *At the best of our knowledge, there is no proof that even by using this super-Lie group as the structure group of a bundle over ordinary spacetime one can get a group of strong bundle automorphisms which equals (or contains) $Diff(M)$.* Hence by gauging the Poincare group one does not get as group of gauge transformations the group of local diffeomorphisms of the spacetime manifold which is the true symmetry group of Einstein's general relativity. The advantage however is that this way nonetheless gives indeed renormalisable theories (in fact "superrenormalisable theories" i.e. spacetime dimension is less than the upper critical dimension), but it also manifests a symmetry between bosons and fermions which does not exist in nature.

Aside: For completeness we would like to mention the following: in *some* cases (namely when the topology of M is such that one can lift a connection of the tangent bundle to a spin connection—this is determined by the second Stiefel-Whitney class of M), one can formulate gravity via the vierbein and the spin connection). But again, the comments made above between $Diff(M)$ and strong tangent bundle automorphisms still apply between $Diff(M)$ and the group of strong bundle automorphisms of the spin bundle (if $\dim M = n$, then the spin bundle has structure group the double cover of $GL(n, \mathbf{R})$ which is the structure group of the tangent bundle TM of M).

We would like to offer some ideas of completely different origin on this approach in the next section.

3 An idea on approximating the group of local diffeomorphisms

Let us start with the elementary fact that given a smooth real function $f : \mathbf{R} \rightarrow \mathbf{R}$, we can use Taylor expansion and approximate f by its derivatives (up to infinite order); let us now assume that f is a smooth map from the smooth manifold M onto itself, where $\dim M = n$; we pick some local coordinates $\{x^i\}$, where $i = 1, 2, \dots, n$ for M ; we know that the tangent bundle TM of M has local coordinates

$$\{x^i, \frac{\partial}{\partial x^i}\}$$

where $\dim(TM) = 2n$, thus schematically TM is like " M plus its first derivative". Similarly, the *tangent bundle of the tangent bundle* $TTM := T^2M$ will have local coordinates containing the x^i 's, their first and second derivatives. Clearly $\dim(T^2M) = 4n$. This is also a bundle over M with structure group $GL(3n; \mathbf{R})$ since, clearly, the composition of projections is again a projection. To approximate a smooth map f from M onto itself then, by imitating the Taylor expansion of a real function of 1 real variable, we need to consider the *infinite order tangent bundle* $T^\infty M$ of M which will be also a bundle over M with structure group $GL(\infty; \mathbf{R})$. [Note: Strictly speaking the principal bundle is the bundle of linear frames of M whose structure Lie group is $GL(n; \mathbf{R})$ and the tangent bundle TM is its associated vector bundle; hopefully there is no misunderstanding caused since we tend not to distinguish between them]. Bundles like $T^n M$ appear in the mathematics literature under the more general title of *jet bundles* (see for instance [4]).

We know that given in general any algebra (or ring) A , we can form the group $GL(\infty; A)$ of invertible $\infty \times \infty$ square matrices with entries from A as follows: we start with $GL(n; A)$ for some finite $n \in \mathbf{N}^*$ (i.e. n is a positive integer); there is a *canonical* way to *inject* $GL(n; A)$ into $GL((n+1); A)$: if $C \in GL(n; A)$ is an $n \times n$ invertible square matrix with entries from the

algebra (or ring) A , we map it onto the following element in $GL((n+1); A)$:

$$C \mapsto \begin{pmatrix} C & 0 \\ 0 & 1 \end{pmatrix}$$

Then we take the *inductive limit* of $GL((n+1); A)$ for $n \rightarrow \infty$ which we denote $GL(\infty; A)$, namely one has

$$GL(\infty; A) = \lim_{n \rightarrow \infty} GL((n+1); A)$$

Let us make one remark before proceeding further: clearly the infinite general linear group will be the corresponding contribution from the infinite order tangent bundle, so similarly to the Taylor expansion, in order to approximate the group of local diffeomorphisms we should "add up" the contributions from all orders of the tangent bundle; yet the final result will be again, in the limit, $GL(\infty; A)$.

There is only one known way to handle this monstrous creature $GL(\infty; A)$, and this is topology: we can define the K-theory groups of A (*due to Bott periodicity we have only two of those*) as follows:

$$\pi_0[GL(\infty; A)] := K^1(A)$$

and

$$\pi_1[GL(\infty; A)] := K^0(A)$$

[Aside:] Using the machinery of the *calculus of functors* (see [19]) in topology, one can indeed think of the homotopy groups as being analogous to the derivatives of a smooth function.

The main point of this argument is that if one wants to *approximate* smooth maps and hence get a grasp on $Diff(M)$, one will probably see K-Theory popping up; this seems reasonable since after all K-theory is an $\infty \times \infty$ generalisation of linear algebra (see for instance [1]).

It is perhaps not clear at this point if one will have to consider $K(M)$, $K(\mathbf{R})$ or its compactification $K(S^1)$. Clearly \mathbf{R} is contractible and non-compact, hence its K-groups are not interesting but we can compactify it to S^1 (this gives a flavour of Kaluza-Klein ideas perhaps) and we know that $K^0(S^1) = \mathbf{Z}$. However the right thing to do, we believe, is to consider $K(M)$ for the following reason: we know that bundles are locally but not

necessarily globally Cartesian products, and hence we want to consider local and not only global gauge transformations to approximate $Diff(M)$; thus the topology of M should be used at some stage. We can be more precise on this point: for convenience we turn from the Lie groups to their corresponding Lie algebras: the Lie algebra b of the Lie group B of local gauge transformations $B = Maps(M \rightarrow G)$ used above can be expressed as $b = g \otimes C(M)$, where g denotes the Lie algebra of the Lie group G and $C(M)$ denotes the (commutative) algebra of functions on the manifold M , namely we consider matrices in g with entries from $C(M)$; that amounts to, in the above discussion, taking $A = C(M)$, (namely we replace \mathbf{R} with $C(M)$ and $GL(n; \mathbf{R})$ with $GL(n; C(M))$ since we need the general linear group as the structure group of the tangent bundle and its powers), hence if we take the inductive limit and then take its fundamental group we shall end up with the K-groups of the algebra $C(M)$; but Serre-Swan theorem tells us that this is equal to the topological K-theory of the manifold M which is what we considered.

The bottom line of this argument is that following the second way (namely try to make the unified theory look like a Yang-Mills theory for which we have a good understanding of quantization and renormalisation), the sought after extended group should be $\tilde{G} = K(M) \times G$, where G is the "honest" Lie group $G = SU(3) \times U(2) \times SO(4)$ for the strong, electroweak and "linear gravity" interactions respectively; yet the "total" group \tilde{G} should contain the semi-direct product with an additional *discrete* group, the K-theory group $K(M)$ of the spacetime manifold M which would take care of the "*nonlinear*" part of the local diffeomorphisms.

It is perhaps more convenient to take the *crossed product noncommutative algebra* $D = K^0(M) \rtimes C(G)$ where $C(G)$ denotes the commutative algebra of functions on the Lie group G seen as a manifold (in which case we are not considering the Lie group structure on G). Since D is a noncommutative algebra, one can very easily turn that into a Lie algebra by taking the commutator of two elements as the Lie bracket. Hence we have a *good candidate at least for the adjoint bundle* of the sought for principal \tilde{G} -bundle. Thus one can define connection 1-forms (gauge potentials) and curvature 2-forms (gauge fields) since these are Lie algebra valued. However this will not be a Lie algebra coming necessarily from a Lie group, at least not in a straightforward way. Perhaps there is an underlying quantum Lie group yet to be discovered. The principal bundle itself, apart from providing the finite

gauge transformations (and not only the infinitesimal ones as the adjoint bundle does), is important for an additional reason: the holonomy of the connection on the \tilde{G} -bundle over spacetime, (which physically corresponds to the Dirac phase factor of the potential which is the true quantum observable from the Aharonov-Bohm effect), is an element of the structure group of the bundle \tilde{G} .

Clearly we end up again with a noncommutative space, since $K(-)$ is a discrete group crossed product with the commutative algebra of functions on an honest Lie group $C(G)$. Hence it appears that whichever of the two obvious approaches one follows for a unified theory of all interactions, noncommutative geometry enters the scene either as a noncommutative space-time or as a noncommutative algebra of some underlying structure group (perhaps a quantum Lie group).

It is fairly clear we believe from the above discussion that this approach involving K-theory gives a better approximation of $Diff(M)$ than supersymmetry, at least topologically.

In order to define the crossed product algebra between the K-group and say $C(G)$, we need an action of the (discrete) group $K^0(M)$ onto G . This can be defined, for example, by using the *holonomy* of a connection, in a similar fashion used in [14], as follows: let P be the total space of a principal G -bundle over M and we assume that M is not simply connected; we know that gauge equivalent classes of flat connections are in one to one correspondence with conjugacy classes of irreducible representations of $\pi_1(M)$ onto G . Let A be a representative of such a class and let us assume that A has holonomy. The holonomy of A defines a map $h : \pi_1(M) \rightarrow G$. Let H denote the image of $\pi_1(M)$ into G under h , namely $H = h(\pi_1(M))$ which is a subgroup of G . Thus we have an action of $\pi_1(M)$ onto G which is defined by the usual multiplication in G restricted to H . But we know that $K^0(M) = \pi_1(GL(\infty; C(M)))$, where $C(M)$ denotes the commutative ring of functions on M . Thus we have an induced "holonomy map". One can relax the flatness condition on the connection A ; in this case, provided that holonomy exists, one still gets a representation of $\pi_1(M)$ onto G but this representation is more complicated and it may not be irreducible. The physical picture is that since we are dealing with gravity where other gauge fields are present, picking a gauge class of connections (potentials), corresponds to a choice of a, say, θ vacuum.

Let us close this section with some remarks:

1. There is a point which is still unclear: why should we take only the fundamental group of $GL(\infty; A)$ and not all its homotopy groups? We know that due to Bott periodicity there are only two K-groups; however one can also use the higher homotopy groups provided one applies Quillen's famous *plus* construction which will give nontrivial higher K-groups. But the role of these higher K-groups is unclear even in the mathematics literature. Another option would be to take the group ring of $\pi_1(M)$ and apply Quillen's ideas to it; this will lead us to the Waldhausen K-theory (see [1]) but it is not easy to relate the Waldhausen K-groups to physics which is what we are trying to do in this article.

2. There are some more versions of "supersymmetric" theories where the Grassmann variables are added to the spacetime manifold as extra degrees of freedom. The philosophy of this approach looks more like the attempt to make the unified theory look like a gravity theory on a "graded commutative" space. Supersymmetry is quite popular in the physics community and since at least until now, there is no experimental evidence for its existence, people assume that it must be spontaneously broken.

3. There has been around in the literature for 15 years or so the notion of *quantum bundles* (through the work of Majid etc, see for example [5]); one can, in a sense, say that what we propose here is some sort of a quantum bundle structure over spacetime where the structure group is not a quantum Lie group (which is what is used in the definition of quantum bundles; a quantum Lie group comes from deformations of classical Lie groups) but it is a noncommutative space (defined via its noncommutative algebra of coordinate functions).

4. Since a Riemannian metric reduces the structure group of the tangent bundle from GL to SO , perhaps one should take KO -groups instead of K -groups.

4 Discussion

Our motivation for this article came from an attempt, eventually, to see if one can say something useful about the problems of renormalisation of quantum gravity, especially under the light of the recent work of Connes and Kreimer (see [6], [9] and [11]). We are not in a position to do that yet but we shall try to make some comments to motivate further research.

Renormalisation has its origin in the Copenhagen interpretation of quantum mechanics as a probabilistic theory; it is absolutely crucial in quantum field theory in order to relate the *bare* quantities of the theory (those are the parameters appearing in the action), with *physical* quantities (those measured in an accelerator during an experiment). The later depend on the energy scale in which each experiment is conducted [thus there is an action of the multiplicative group \mathbf{R}_+ on the space of all physical parameters; this space is often a manifold and the group action is not always free, it may have fixed points]; moreover many of them appear to be infinite; in many cases this is inherited by the classical theory (e.g. like the charge of the electron in QED: it is infinite classically if we consider the electron as a point particle with the inverse square law and it remains infinite in QED prior to renormalisation); in fact renormalisation at first it was considered as an ingenious (but rather *ad hoc*) way to extract finite answers by using a *finite* supply of counterterms which cancel the infinities (divergent integrals) appearing in the perturbation series expansion of the action using Feynman graphs, and at the same time all quantities become independent from the energy scale at the end of the calculation. In all this, gauge symmetry (via Ward-Takahasi and Slavnov-Taylor equations for the abelian and non-abelian cases respectively) plays a key role. Recent developments related to the renormalisation group flow however have pointed out that renormalisation is something really deep in physics and it represents a lot more than a number of clever techniques (eg the qualitative explanation of the finiteness of electric charge of the electron by "screening", uncertainty principle and polarisation of the vacuum).

It has been observed that if the Lagrangian contains combinations of field operators of excessively high dimension in energy units compared with the spacetime dimension, the counterterms required to cancel all divergencies proliferate to *infinite* number, and, at first glance, the theory would seem to gain an infinite number of free parameters and therefore it loses all

predictive power. This does not happen for electroweak and strong interactions (although in general this is not the end of the story since there may be anomalies), but it seems to happen in gravity if one applies perturbation theory in it. In such a case, there are very few escapes if one is lucky: gauge symmetry (that is why we care so much about the gauge group of symmetries), the so-called *Fisher-Wilson fixed points* in the renormalisation group flow or nonperturbative effects.

Before elaborating more on these Fisher-Wilson fixed points, let us try to recall the current level of affairs about perturbative quantum gravity (see [16] and [17]) (but at the same time we shall make a more general study): we focus on Newton's constant G ; in general, it depends on the energy scale; using physical units as usual we translate length into the inverse of momentum. This allows one to think G as a function of momentum p , namely $G = G(p)$ (higher momentum means shorter distances). The equation describing how G depends on p is the *Callan-Symanzik* equation:

$$\frac{dG}{d(\ln p)} = \beta(G)$$

which involves the β -function of our theory. Typically $\beta(G) = (n - d)G + aG^2 + bG^3 + \dots$ where n is spacetime dimension and d is the *upper critical dimension* (this is the spacetime dimension in which the coupling constant is dimensionless). Let's take the linear term:

$$\frac{dG}{d(\ln p)} = (n - d)G$$

which means that G is proportional to p^{n-d} . One can study 3 cases:

- (i) if $n < d$, then the Newtonian constant gets *smaller* at higher momenta which means that in higher energy scales there is practically no interaction and the theory is essentially *free* (superrenormalisable theory);
- (ii) if $n > d$, then Newton's constant gets larger at higher energies, so gravity becomes very strong (in fact infinitely strong) as momentum increases. In this case the theory is (superficially perturbatively) nonrenormalisable; things are very difficult but there may be some hope in some cases;
- (iii) if $n=d$, we say the theory is renormalisable (like QED) but we have to calculate the next term in the β -function.

We are interested in quantum gravity in 4 dimensions, namely $n = 4$. So we have to figure out what d is. Well, a not very hard argument about

dimension and units says that $d = 2$ (see for example [16], whereas the upper critical dimension for Yang-Mills theories is 4). So gravity in 4 dimensions is (superficially at least) nonrenormalisable. Yet there is a subtlety here: we should take care of the higher order terms of the β -function.

Let's focus on the second term; this becomes dominant if $n = d$, ie for renormalisable theories (like QED):

$$\frac{dG}{d(\ln p)} = aG^2.$$

One can solve this easily and get

$$G = \frac{c}{1 - ac \ln p}$$

where c is a positive constant. If $a < 0$ the coupling constant slowly decreases with increasing momentum, in this case we say that our theory is *asymptotically free* (like QCD).

If however $a > 0$, then G increases as the energy scale goes up, in fact it becomes infinite at sufficiently high energies, this is called a *Landau pole* (this happens in QED when we do not include the weak force). Among renormalisable theories the ones with $a < 0$ are considered "good" and the ones with $a > 0$ are considered "bad" (like QED).

Let's come to gravity now: perturbative quantum gravity in dim 2 is not only renormalisable, in fact it is asymptotically free, since for gravity $a < 0$. If we ignore higher order terms, this implies something very interesting about gravity in 4 dimensions: if we see only the first 2 terms, then G increases as the momentum increases (as an honest nonrenormalisable theory would do) but when G gets big enough, the second term matters more (remember it has a negative coefficient in front of it) and thus after a while the growth of G starts slowing!

There is *strong numerical evidence* (see [17]) that in fact

$$\lim_{p \rightarrow \infty} \frac{dG}{d(\ln p)} \rightarrow 0$$

This is called an "*ultraviolet stable fixed point*". Mathematically, it attracts nearby points as we flow in the direction of higher momenta. This particular kind of ultraviolet stable fixed point-coming from an asymptotically free

theory in dimensions above its upper critical dimension-is called a "*Fisher-Wilson*" *fixed point* (see [16]).

In general β -function computations are hard. The big question then is the following: can we use the new Connes-Kreimer approach to β -function using the Hopf algebra of Feynman graphs (or equivalently the Hopf algebra of decorated rooted trees, see [6], [9] and [11]) in order to prove that the β -function of perturbative quantum gravity has a Fisher-Wilson fixed point?

5 Appendix

Let us for convenience, describe briefly the Connes-Kreimer approach to renormalisation; *given a specific quantum field theory*, namely its action and its symmetries, (although in [10] the author managed to describe a generic quantum Yang-Mills theory purely combinatorial without starting from a gauge invariant Yang-Mills type of action), we need the following data (in momentum space, a similar description exists in coordinate space): (H, V, R, ϕ) where H is the Hopf algebra of Feynman graphs (equivalently one can consider the Hopf algebra of decorated rooted trees), V is the regularisation algebra (for dimensional regularisation $V = \mathbf{C}[\epsilon^{-1}, \epsilon]$, ie V is the algebra of Laurent series with finite pole part), $\phi : H \rightarrow V$ is a (umital) algebra homomorphism, the "regularisation map" which can be seen as corresponding to a choice of a boundary condition for the Dyson-Schwinger equation (regularised Feynman rules, see [11]) and $R : V \rightarrow V$ is the "renormalisation scheme": it satisfies the Rota-Baxter equation (this equation provides the link between renormalisation and the Birkhoff decomposition from which one can get the Riemann-Hilbert correspondence), it has $R(1) = 1$ and it preserves the UV divergent structure (ie the pole part-e.g. in the minimal subtraction scheme, R is the projector onto the proper part). The Hopf algebra H has a coproduct Δ (which disentangles trees and divergencies to subtrees and subdivergencies) and an antipode S . One then defines the twisted antipode $S_R^\phi : H \rightarrow V$ which provides the relevant counterterm and the convolution $S_R^\phi * \phi = \phi_R$ which solves the Bogoliubov recursion. The Hopf algebra structure can be determined by the perturbative expansion of the action into Feynman graphs; one can have two models for H : either the graded, free, commutative algebra generated by trees (by a tree we mean a connected, contractible, compact graph) with the weight (which is the

number of vertices) grading, or, given a set S , the graded, free, commutative algebra generated by S -decorated trees (we decorate only the vertices and not the edges of the graphs). All Hopf algebras appearing in various quantum field theories are Hopf subalgebras of the two models above. Note that we can base the Hopf algebra on 1PI (1 particle irreducible) Feynman graphs instead of trees, these are equivalent descriptions. We are interested in the Hochschild cohomology of the Hopf algebra of trees H , in fact since Hochschild cohomology groups vanish in dimension greater than 2, we are only interested in the first Hochschild cohomology groups. This is crucial, since in the Connes-Kreimer framework, the requirement of the locality of the counterterms in renormalisation is interpreted as linear functionals on H being b -closed, where b is the Hochschild differential (see [10] and [11]).

To the Hopf algebra of graphs H one can associate a Lie algebra and a Lie group, let's denote it G , using the Milnor-Moore theorem. Roughly G comes from the group of characters of H . The antipode map S in the Hopf algebra delivers the same terms as those needed for the subtraction procedure in renormalisation. One can understand S by using the Riemann-Hilbert correspondence.

A well-known instance of the Riemann-Hilbert correspondence (which in general gives equivalences between geometric problems associated with differential systems with singularities and representation theoretic data) is the 1:1 correspondence between (gauge equivalence classes of) *flat* connections on a vector bundle and (conjugacy classes of) representations of the fundamental group of the base manifold onto the structure Lie group, the correspondence given by the *holonomy* of the flat connection.

Essentially Connes-Kreimer used a variation of the above well-known example of the Riemann-Hilbert correspondence: the base is an infinitesimal punctured disc (which is a non-simply connected space) Δ^* of \mathbf{CP}^1 around the point $D = 4$ (the complex surface comes from the complexification of dimension which from \mathbf{N} takes values in \mathbf{R} according to the rules of dimensional regularisation where dimension becomes real and then we complexify), the structure group is \mathbf{G}_m which is just the complex numbers (as multiplicative group) and the total space is denoted B . The fibre represents *rescaling* and then we study equisingular (representing independence of choice of unit of mass) G -valued flat connections (the Lie group G is the one coming from group of characters of the Hopf algebra H of Feynman

graphs due to Kreimer). From the representation theory side we study representations $U^* \rightarrow G^*$ where U is the universal group of all physical theories and $G^* = G \rtimes_{\theta} \mathbf{R}$ (see [8]).

Now let us try to duplicate the Connes-Kreimer framework in the case of gravity, namely we would like to build the Riemann-Hilbert correspondence for quantum gravity. Clearly U^* is a universal group for all physical theories, hence there is no change here. Similarly the punctured disc Δ^* should remain the same for a possible dimensional regularisation of quantum gravity in dim 4. The structure group \mathbf{G}_m which represents rescaling should not be changed either. Thus the only ingredient which changes for the case of quantum gravity is the Connes-Kreimer group G since gravity has different action and different group of gauge transformations.

We would like to finish this section with the following remark: if we manage to prove that the β -function of Newton's constant for quantum gravity has a Fisher-Wilson fixed point, (using the Connes-Kreimer framework or other), that would definitely be a major breakthrough. Yet this would not be the end of the story since a proper quantum theory of gravity should explain the cosmological constant puzzles as well. Moreover, quantum gravity is a theory which has also, crucially, *nonlocal features* (e.g. holography principle); so relying solely on the Hochschild cohomology of the Hopf algebra is too restrictive; somehow, following the line of argument in [12], [13] and the Connes-Kreimer approach to renormalisation, this may be related to operads and the action of the Grothendieck-Teichmüller group on the Hochschild complex. But there is another feature that a possible quantum gravity theory must have: it must be *asymmetric in time* in order to explain the difference in entropy between the big-bang and the big-crunch (see [18]). Unfortunately both general relativity and quantum mechanics are theories which are symmetric in time. That, perhaps, would require a more radical approach to quantum gravity than simply trying to immitate perturbative quantum field theory ideas.

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